Signed degree sets in signed graphs

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Abstract

The set D of distinct signed degrees of the vertices in a signed graph G is called its signed degree set . In this paper, we prove that every non-empty set of positive (negative) integers is the signed degree set of some connected signed graph and determine the smallest possible order for such a signed graph . We also prove that every non-empty set of integers is the signed degree set of some connected signed graph .

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1. Introduction

All graphs in this paper are finite , undirected , without loops and multiple edges . A signed graph G is a graph in which each edge is assigned a positive or a negative sign . These were first discovered by Harary [3] . The signed degree of a vertex v_i in a signed graph G is denoted by sdeg(v_i)(or simply by d_i)and is defined as the number of positive edges incident with $v_i$less the number of negative edges incident with v_i.So, if$v_i$is incident with d_i^+positive edges and d_i^- negative edges, then sdeg(v_i)= d_i^+-d_i^-.A signed degree sequence $\sigma = [d_1, d_2, ..., d_n]$of a signed graph G is formed by listing the vertex signed degrees in non-increasing order . A sequence $\sigma = [d_1, d_2, ..., d_n$integers is graphical if σ is a signed degree sequence of some signed graph . Also, a non-zero sequence $\sigma = [d_1, d_2, ..., d_n$is a standard sequence if σ is non-increasing ,$\sum_{i=1}^{n} d_i$ is even,$d_1 > 0$,each $|d_i| < n$, and $|d_1| \geq |d_n|$.

The following result , due to Chartrand et al. [1] , gives a necessary and sufficient condition for a sequence of integers to be graphical , which is similar to Hakimi's result for degree sequences [2] .

Theorem 1.1. Let $\sigma = [d_1, d_2, ..., d_n]$ be a standard sequence . Then , s is graphical if and only if there exist integers r and s with $d_1 = r - s$ and $0 \leq s \leq \frac{n-1-d_1}{2}$such that

$$\sigma^{/} = [d_2 - 1, d_3 - 1, ..., d_{r+1} - 1, d_{r+2}, d_{r+3}, ..., d_{n-s}, d_{n-s+1} + 1, ..., d_n + 1,]$$

is graphical .

The next characterization for signed degrees in signed graphs is given by Yan et al. [5] .

Theorem 1.2. A standard integral sequence $\sigma = [d_1, d_2, ..., d_n]$ is graphical if and only if

$$\sigma_m^{/} = [d_2 - 1, ..., d_{d_1+m+1} - 1, d_{d_1+m+2}, ..., d_{n-m}, d_{n-m+1} + 1, ..., d_n + 1,]$$

is graphical , where m is the maximum non-negative integer such that $d_{d_1+m+1} > d_{n-m+1}$

In [4], Kapoor et al. proved that every non-empty set of distinct positive integers is the degree set of a connected graph and determined the smallest order for such a graph .

2. Main Results

First we have the following definition .
Definition. The set D of distinct signed degrees of the vertices in a signed graph G is called its signed degree set .

Now, we obtain the following results .

Theorem 2.1. Every non-empty set D of positive integers is the signed degree set of some connected signed graph and the minimum order of such a signed graph is N + 1, where N is the maximum integer in the set D.
Proof. Let D be a signed degree set and n_0(D) denotes the minimum order of a signed graph

G realizing D. Since N is the maximum integer in D, therefore there is a vertex in G which is adjacent to at least N other vertices with a positive sign. Then,$n_0(D) \geq N + 1$. Now, if there exists a signed graph of order N + 1 with D as signed degree set , then $n_0(D)$ = N + 1. The existence of such a signed graph is obtained by using induction on the number of elements of D.

Let $D = \{d_1, d_2, \ . \ . \ . \ , d_n\}$, where $d_1 < d_2 < \ . \ . \ . \ < d_n$, be a set of positive integers . For n = 1, let G be a complete graph on $d_1 + 1$ vertices , that is K_{d_1+1} in which each edge is assigned a positive sign. Then ,

$$sdeg(v) = (d_1 + 1 - 1) - 0 = d_1, for \quad all \quad v \in V(G)$$

Therefore , G is a signed graph with signed degree set D = $\{d_1\}$.

For n = 2 , let G_1 be a complete graph on d_1 vertices, that is K_{d_1},in which each edge is assigned a positive sign and let G_2 be a null graph on $d_2 - d_1 + 1 > 0$ vertices , that is $K_{d_2-d_1+1}$ Join every vertex of G_1 to each vertex of G_2 with a positive edge , so that we obtain a signed graph G on $d_1 + d_2 - d_1 + 1 = d_2 + 1$ vertices with

$$sdeg(u) = (d_1 - 1) + (d_2 - d_1 + 1) - 0 = d_2, for \quad all \quad u \in V(G_1),$$

and

$$sdeg(v) = (0) + (d_1) - 0 = d_1, for \quad all \quad v \in V(G_2)$$

.
Therefore , signed degree set of G is D = d_1, d_2.

For n = 3, let G_1 be a complete graph on d_1 vertices, that is K_{d_1} , in which each edge is assigned a positive sign, G_2 be a complete graph on $d_2 - d_1 + 1 > 0$ vertices , that is $K_{d_2-d_1+1}$ in which each edge is assigned a positive sign, and G_3 be a null graph on $d_3 - d_2 > 0$ vertices, that is $\bar{K}_{d_3-d_2}$ Join every vertex of G_1 to each vertex of G_2 with a positive edge and join every vertex of G_1 to each vertex of G_3 with a positive edge, so that we obtain a signed graph G on $d_1 + d_2 - d_1 + 1 + d_3 - d_2 = d_3 + 1$ vertices with

$$sdeg(u) = (d_1 - 1) + (d_2 - d_1 + 1) + (d_3 - d_2) - 0 = d_3, for \quad all \quad u \in V(G_1),$$

$$sdeg(v) = (d_2 - d_1 + 1 - 1) + (d_1) - 0 = d_2, for \quad all \quad v \in V(G_2),$$

and

$$sdeg(w) = (0) + (d_1) - 0 = d_1, for \quad all \quad w \in V(G_3).$$

Therefore , signed degree set of G is D = $\{d_1, d_2, d_3\}$.

Assume that the result holds for k . We show that the result is true for k + 1.

Let D = $\{d_1, d_2, \ . \ . \ . \ , d_k, d_{k+1}\}$ be a k + 1 set of positive integers with $d_1 < d_2 < \ . \ . \ . \ < d_k < d_{k+1}$. Clearly , $0 < d_2 - d_1 < d_3 - d_1 < \ . \ . \ . \ < d_k - d_1$. Therefore , by induction hypothesis ,

there is a signed graph G_1 realizing the signed degree set $D_1 = \{d_2 - d_1, d_3 - d_1, \ . \ . \ . \ , d_k - d_1\}$ on $d_k - d_1 + 1$ vertices as $|V(D_1)| < k$. Let G_2 be a complete graph on d_1 vertices, that is K_{d_1} , in which each edge is assigned a positive sign and G_3 be a null graph on $d_{k+1} - d_k > 0$ vertices, that is $\bar{K}_{d_{k+1}-d_k}$. Join every vertex of G_2 to each vertex of G_1 with a positive edge and join every vertex of G_2 to each vertex of G_3 with a positive edge, so that we obtain a signed graph G on $d_k - d_1 + 1 + d_1 + d_{k+1} - d_k = d_{k+1} + 1$ vertices with

$$sdeg(u) = (d_i - d_1) + (d_1) - 0 = d_i, for \quad all \quad u \in V(G_1) \quad where \quad 2 \leq i \leq k,$$

$$sdeg(v) = (d_1 - 1) + (d_k - d_1 + 1) + (d_{k+1} - d_k) - 0 = d_{k+1}, for \quad all \quad v \in V(G_2),$$

and

$$sdeg(w) = (0) + (d_1) - 0 = d_1, for \quad all \quad w \in V(G_3).$$

Therefore , signed degree set of G is D = $d_1, d_2, \ . \ . \ . \ , d_k, d_{k+1}$. Clearly , by construction , all the signed graphs are connected . Hence, the result follows .

Theorem 2.2. Every non-empty set D of negative integers is the signed degree set of some connected signed graph and the minimum order of such a graph is $|M| + 1$, where M is the minimum integer in the set D .

Proof. . Let D be a signed degree set and $m_0(D)$ denotes the minimum order of a signed graph G realizing D . Since $|M|$ is the maximum integer in D , therefore there is a vertex in G which is adjacent to at least $|M|$ other vertices with a negative sign . Then ,$m_0(D) \geq |M| + 1$. Now , if there exists a signed graph of order $|M|+1$ with D as signed degree set , then $m_0(D) = |M|+1$.

Let D $= \{-d_1, -d_2, \ . \ . \ . \ , -d_n\}$, $\quad -d_1 > -d_2 > \ . \ . \ . \ > -d_n$, be a set of negative integers where $d_1, d_2, , d_n$ are positive integers . Now , $D_1 = \{d_1, d_2, \ . \ . \ . \ , d_n\}$ is a set of positive integers with $d_1 < d_2 < . < d_n$. By Theorem 2.1 , there exists a connected signed graph G_1 on $d_n + 1 = |-d_n| + 1$ vertices with signed degree set $D_1 = \{d_1, d_2, \ . \ . \ . \ , d_n\}$. Now , construct a signed graph G from G_1 by interchanging positive edges with negative edges . Then , G is a connected signed graph on $|-d_n| + 1$ vertices with degree set D $= \{-d_1, -d_2, \ . \ . \ . \ , -d_n\}$. This proves the result .

Theorem 2.3. Every non-empty set D of integers is the signed degree set of some connected signed graph .

Proof. Let D be a set of n integers . We have the following cases .

Case I. D is a set of positive (negative) integers. Then , the result follows by Theorem 2.1(Theorem 2.2) .

Case II. D = $\{0\}$. Then , a null graph G on one vertex, that is K_1, has signed degree set D = $\{0\}$.

Case III. D is a set of non-negative (non-positive) integers. Let $D = D_1 \cup \{0\}$, where D_1 is a set of positive (negative) integers. Then , by Theorem 2.1 (Theorem 2.2) , there is a signed graph G_1 with degree set D_1 . Let G_2 be a null graph on two vertices , that is $\bar{K}_2$. Let e = uv be an edge in G_1 with positive (negative) sign and let $x, y \in V(G_2)$. Add the positive (negative) edges ux and vy , and the negative (positive) edges uy and vx , so that we obtain a connected signed graph G with signed degree set D . We note that addition of such edges do not effect the

signed degrees of the vertices of G_1 , and the vertices x and y have signed degrees zero each .

Case IV. D is a set of non-zero integers . Let $D = D_1 \cup D_2$, where D_1 is a set of positive integers and D_2 is a set of negative integers . Then , by Theorem 2.1 and Theorem 2.2 , there are connected signed graphs G_1 and G_2 with signed degree sets D_1 and D_2 . Let $e_1 = uv$ be an edge in G_1 with positive sign and $e_2 = xy$ be an edge in G_2 with negative sign . Add the positive edges ux and vy , and the negative edges uy and vx , so that we obtain a connected signed graph G with signed degree set D . We note that addition of such edges do not effect the signed degrees of the vertices of G_1 and G_2 .

Case V. D is a set of integers . Let $D = D_1 \cup D_2 \cup \{0\}$, where D_1 and D_2 are the sets of positive and negative integers respectively . Then, by Theorem 2.1 and Theorem 2.2 , there are connected signed graphs G_1 and G_2 with signed degree sets D_1 and D_2 . Let G_3 be a null graph on one vertex , that is K_1 . Let $e_1 = uv$ be an edge in G_1 with positive sign , and let $x \in V(G_2)$ and $y \in V(G_3)$. Add the positive edges uy and vx , and the negative edges ux and vy , so that we obtain a connected signed graph G with signed degree set D. We note that addition of such edges do not effect the signed degrees of the vertices of G_1 and G_2 , and the vertex y has signed degree zero . This completes the proof .

Theorem 2.4.. If G is a signed graph with vertex set V where $|V| = r$ and signed degree set $\{d_1, d_2, \ . \ . \ . \ , d_n\}$. Then , for each $k \geq 1$, there is a signed graph with kr vertices and signed degree set $\{d_1, d_2, \ . \ . \ . \ , d_n\}$.

Proof. For each i , $1 \leq i \leq k$, let G_i be a copy of G with vertex set V_i . Define a signed graph H with vertex set $W = \cup_{i=1}^{k} v_i$ where $V_i \cap V_j = \phi (i \neq j)$ and the edges of H are the edges of G_i for all i, where $1 \leq i \leq k$. Therefore , H is a signed graph on kr vertices with signed degree set $\{d_1, d_2, \ . \ . \ . \ , d_n\}$.

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